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One-dimensional plastic materials with work-hardening

II. Acceleration waves

T. TOKUOKA

Department of Aeronautical Engineering, Kyoto University, Kyoto, Japan

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SUMMARY

Acceleration waves in one-dimensional plastic materials are investigated by the theory of singular points. The unloading wave propagates with a constant velocity, while the propagation velocity of the loading wave is less than that of the unloading wave and the velocity depends upon the stress and the work-hardening. The growth and decay of the amplitude of the waves are also analyzed. The unloading wave propagates with a constant amplitude. The amplitude of the loading wave may grow or decay and the choice between the two depends upon the stress, the work-hardening and whether the wave is compressive or expansive. In the case of growth the amplitude tends to infinity in finite time, that is, the blow time, and the acceleration wave coalesces into a shock wave. In the case of decay the amplitude tends to zero as the time tends to infinity. The propagation velocity, the blow time and the blow distance are calculated and plotted against the strain.

1. Introduction

Wave propagation in a material depends upon the character of the material and we can estimate its mechanical properties by means of the analysis of the behavior of the wave.

Until now there has been done a lot of theoretical work on wave propagation in any type of material. Usually a wave is defined to be a *singular surface* and an *acceleration wave* is defined to be a surface across which a jump in the acceleration of the material particle occurs. For singular surfaces we refer to, *e.g.,* Truesdell and Toupin [1].

Acceleration waves in *hypo-elastic materials* were analyzed by Bernstein [2], Hill [3], Truesdell [4] and Varley and Dunwoody [5], and those in rate-type *plastic materials* were studied by the author [6] – [8]. For the reason that the analysis is simple without much loss of generality, waves in one-dimensional materials have been investigated frequently. For a one-dimensional simple material, we refer to [9].

In the preceding article [10] we studied a *one-dimensional plastic material with general work-hardening* and investigated its loading and unloading behavior. In this paper we shall study acceleration waves in such a material. The propagation velocities are obtained and the growth and the decay of the waves are analyzed for the *unloading* and the *loading wave.*

2. Field equations and the constitutive equations

The one-dimensional *motion* of a material particle is given by

$$x = \chi(X, t), \tag{2.1}$$

where X and x are, respectively, the coordinates of a material particle in the *reference* and the *current configuration* and t is the time. Every one-dimensional material must satisfy the *equation of motion*

$$\frac{\partial T}{\partial X} + \rho_R b = \rho_R \ddot{x}, \tag{2.2}$$

where T is the stress, $\ddot{x}$ is the acceleration, ρ_R is the mass density, b is the body force and ρ_R and b refer to the reference configuration. The *conservation law of mass* is expressed as

$$\dot{\rho} + \rho \frac{\partial \dot{x}}{\partial x} = 0. \tag{2.3}$$

The *deformation gradient* and the *strain* are defined, respectively, by

$$F = \frac{\partial \chi}{\partial X}, \qquad e = F - 1. \tag{2.4}$$

The *constitutive equations* of a plastic material with general work-hardening were proposed in the preceding paper [10]. They are

$$\frac{d\tilde{T}}{de} = K(\tilde{T}, \alpha), \qquad \frac{d\alpha}{de} = \Phi(\tilde{T}) \qquad \frac{d\beta}{de} = \Psi(\tilde{T}, \alpha), \tag{2.5}$$

where the *tangential moduli* are defined by

$$K = \lambda, \qquad \Phi = 0, \qquad \Psi = 0, \tag{2.6}$$

in the unloading state, and by

$$K = \lambda - \frac{\tilde{T}^2}{\lambda M(\alpha)^2}, \qquad \Phi = \frac{\tilde{T}}{\lambda}, \qquad \Psi = \frac{c\tilde{T}^2}{\lambda M(\alpha)^2}, \tag{2.7}$$

in the loading state;

$$\tilde{T} \equiv T - \beta \tag{2.8}$$

is the *translated stress*, α and β are internal state variables and called, respectively, the *parameter* *of the isotropic work-hardening* and the *translation*, $M(\alpha)$ is a material function of α, and λ and c are positive material constants. When the rate of work done by the stress is given by

$$w = T\frac{\dot{e}}{F}, \tag{2.9}$$

the *unloading* and the *loading* state are defined, respectively, by the conditions

$$w < 0, \qquad\qquad w \geqq 0. \tag{2.10}$$

The *yield condition* is defined by the vanishing of the translated stress modulus with respect to the strain and it is given by

$$\widetilde{T} = \pm\,\lambda\, M(\alpha). \tag{2.11}$$

The yield value of the translated stress was discussed in the preceding article and its second-order approximation has the value

$$\widetilde{T} = \pm\,\lambda\, M(\alpha)\;(1 - M(\alpha)\;M(\alpha)')^{\frac{1}{2}}. \tag{2.12}$$

Now we assume that

$$M(\alpha) > 0, \qquad\qquad M(\alpha)' \geqq 0, \tag{2.13}$$

$$0 \leqq c \leqq 1. \tag{2.14}$$

The inequality $(2.13)_2$ denotes the *work-hardening*. If $c > 1$, the stress can not yield although the translated stress does, so we imposed restriction (2.14) on c.

3. Singular points, compatibility conditions and acceleration waves

In a one-dimensional space a *singular point* W is expressed by

$$X = Z(t), \qquad\qquad x = z(t), \tag{3.1}$$

where the functions Z and z are related by the motion (2.1) such that

$$z(t) = \chi(Z(t), t). \tag{3.2}$$

The propagation velocity of the point W is given by

$$U = \dot{Z}(t), \qquad\qquad u = \dot{z}(t), \tag{3.3}$$

which refer to the reference and the current configuration, respectively, and from $\dot{z} = \partial\chi/\partial t + (\partial\chi/\partial X)\dot{Z}$ we have

$$u - \dot{x} = F\,U, \tag{3.4}$$

which denotes the velocity of W relative to the material particle. Henceforth we assume

$$U > 0. \tag{3.5}$$

A quantity ψ and its derivatives are assumed to be continuous everywhere except at W, but they may have jumps across W. The *jump* of ψ is defined by

$$[\psi] = \psi^- - \psi^+, \tag{3.6}$$

where ψ^- and ψ^+ are the limiting values from the negative and the positive side of W, respectively.

The jumps of a quantity and of its derivatives can not attain arbitrary values but they must satisfy the *compatibility conditions*

$$[\dot{\psi}] = -\,U B + \frac{\delta A}{\delta t}, \tag{3.7}$$

$$\left[\frac{\partial\dot{\psi}}{\partial x}\right] = -\,U C + \frac{\delta B}{\delta t}, \tag{3.8}$$

$$\left[\ddot{\psi}\right] = U^2\,C - 2U\,\frac{\delta B}{\delta t} - \frac{\delta U}{\delta t}B + \frac{\delta^2 A}{\delta t^2}, \tag{3.9}$$

where

$$A = [\psi], \qquad B = \left[\frac{\partial\psi}{\partial X}\right], \qquad C = \left[\frac{\partial^2\psi}{\partial X^2}\right] \tag{3.10}$$

and the *displacement derivative* $\delta\phi/\delta t$ of any quantity ϕ means the time rate of ϕ measured by an observer moving with the singular point. For these conditions see, *e.g.*, Chen [11].

An *acceleration wave* in a plastic material having the constitutive equations (2.5) – (2.7) is defined to be a singular point, across which x, $\dot{x}$, e, T, α and β are continuous but $\ddot{x}$ has a jump

$$a = [\ddot{x}], \tag{3.11}$$

which is called the *amplitude* of the wave.

From the above definition,

$$[x] = [\dot{x}] = [e] = [T] = [\alpha] = [\beta] = 0 \tag{3.12}$$

and $a \neq 0$. Substituting x into ψ in (3.8) and (3.9), we have

$$[e] = -\frac{a}{U}, \qquad \left[\frac{\partial e}{\partial x}\right] = \frac{a}{U^2}. \tag{3.13}$$

The conservation law of mass (2.3) is expressed as

$$\dot{\rho} = -\rho\frac{\dot{e}}{F}, \qquad \rho F = \rho_R. \tag{3.14}$$

Then from $(3.12)_2$ and $[\rho] = 0$ we have

$$[\dot{\rho}] = \frac{\rho a}{FU}. \tag{3.15}$$

The deformation is assumed to be non-singular, so $F > 0$. From the assumption (3.5), the inequality $\rho > 0$ and the definition of a jump (3.6) we can say that, when $a > 0$, then $[\dot{\rho}] > 0$ and the singular point represents a *compressive wave*; when $a < 0$, then $[\dot{\rho}] < 0$ and it represents an *expansive wave.*

4. Propagation velocities

The equation of motion (2.2) holds except for the singular point. Then we can take the jump of it across W and we have

$$\left[\frac{\partial T}{\partial X}\right] = \rho_R a, \tag{4.1}$$

where the body force is assumed to be continuous. The constitutive equations of the stress $(2.5)_1$ can be written as

$$\frac{\partial T}{\partial X} = \lambda\frac{\partial e}{\partial X}, \qquad \frac{\partial T}{\partial X} = \left\{\lambda - \frac{(1-c)\tilde{T}^2}{\lambda M(\alpha)^2}\right\}\frac{\partial e}{\partial X} \tag{4.2}$$

in the unloading state and in the loading state, respectively.

Now we assume that both regions ahead and behind of W are in the unloading state or in the loading state. For the former case the wave is called an *unloading wave* and for the latter case it is called a *loading wave*. Then we have

$$\left[\frac{\partial T}{\partial X}\right] = \frac{\lambda}{U^2}a, \qquad \left[\frac{\partial T}{\partial X}\right] = \left\{1 - \frac{(1-c)\tilde{S}^2}{M(\alpha)^2}\right\}\frac{\lambda}{U^2}a, \tag{4.3}$$

where

$$\tilde{S} \equiv \frac{\tilde{T}}{\lambda} = S - \gamma, \qquad S \equiv \frac{T}{\lambda}, \qquad \gamma \equiv \frac{\beta}{\lambda} \tag{4.4}$$

are non-dimensional quantities. Therefore, substituting (4.3) into (4.1), we have the *propagation velocity* of the unloading wave

$$U_E = \left(\frac{\lambda}{\rho_R}\right)^{\frac{1}{2}}, \tag{4.5}$$

and that of the loading wave

$$U = U_E \left\{1 - \frac{(1-c)S^2}{M(\alpha)^2}\right\}^{\frac{1}{2}}. \tag{4.6}$$

For a material with $c = 1$ the loading wave propagates with U_E.

In the yield state the translated stress has the value (2.12). Then the loading wave propagates with the velocity

$$U = U_E \{c + (1-c) M(\alpha) M(\alpha)'\}^{\frac{1}{2}} \tag{4.7}$$

in the yield state. For a material without translational work-hardening it is

$$U = U_E \, (M(\alpha) M(\alpha)')^{\frac{1}{2}}. \tag{4.8}$$

For a material without isotropic work-hardening it is

$$U = U_E \, c^{\frac{1}{2}}. \tag{4.9}$$

For a *perfectly plastic material* with $a = c = 0$, the loading wave can not propagate in the yield state.

In the preceding article [10] we calculated stress-strain relations of a material which has the material function

$$M(\alpha) = M_0 \, (1 + a\alpha)^n, \tag{4.10}$$

where the material was loaded from the initial state

$$\tilde{T} = 0, \qquad \alpha = 0, \qquad \beta = 0. \tag{4.11}$$

Figure 1 shows the non-dimensional propagation velocities U/U_E of the loading waves against the strain in the loading states which were presented in Figure 2 of [10]. The velocities decrease when the states approach the yield state and tend to the values (4.7) – (4.9) which are shown by the black circles plotted at $e = 4 \times 10^{-3}$. For materials which have a small value of the material constant c, the loading wave propagates with small velocity, and for the material with $c = 1$ the loading wave propagates with U_E at any state.

5. Growth and decay

Let us study the growth and decay of the amplitude of the acceleration waves. Differentiating the equation of motion (2.2) with time we have

$$\frac{\partial \dot{T}}{\partial X} + \rho_R \dot{b} = \rho_R \dddot{x}, \tag{5.1}$$

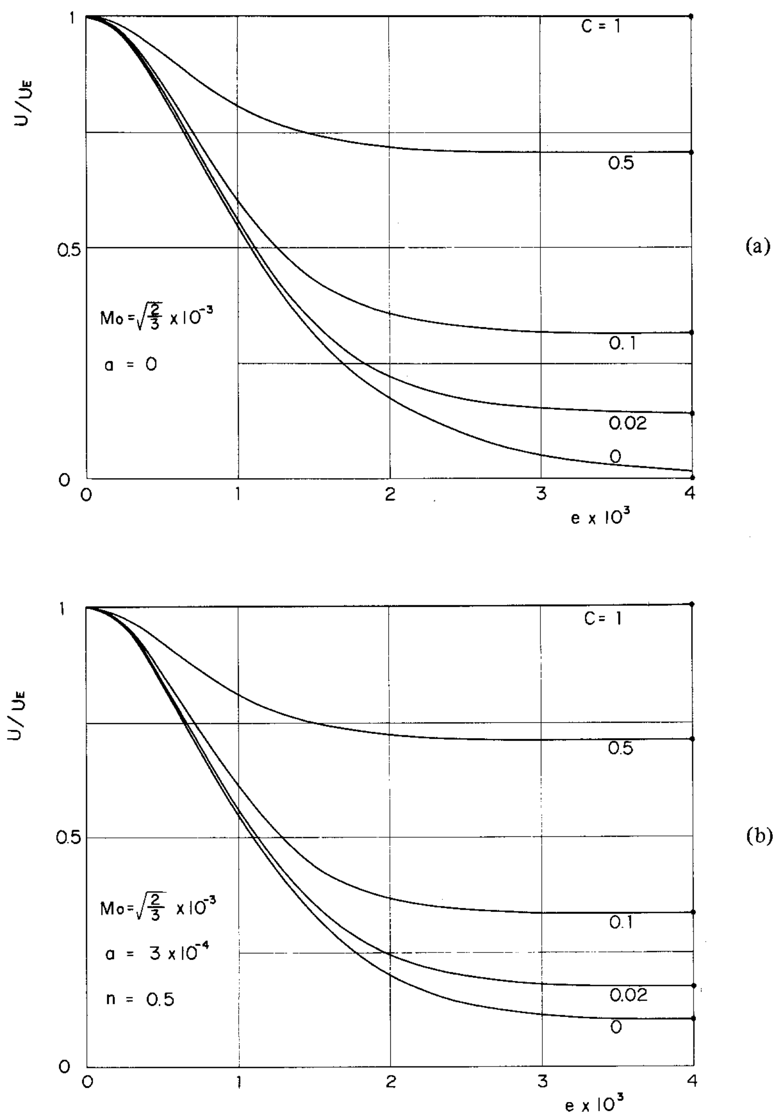


Figure 1. Non-dimensional propagation velocities of the loading wave against the strain in the loading state. Black circles denote the limit velocities at the yield state.

which holds except for W. Then we have

$$\left[\frac{\partial \dot{T}}{\partial X}\right] = \rho_R \, [\ddot{x}\,] \tag{5.2}$$

under the condition $[\dot{b}\,] = 0$.

Now we assume that the material in the region ahead of the wave is in the homogeneous equilibrium state. So the propagation velocity is constant and $\delta U/\delta t = 0$. Putting $\psi = \dot{x}$ in the compatibility condition (3.9) and using $(3.13)_1$, we have

$$[\,\ddot{x}\,] = U^2 \left[\frac{\partial \dot{e}}{\partial X}\right] + 2\frac{\delta a}{\delta t}. \tag{5.3}$$

From the constitutive equation for the stress we have

$$\frac{\partial \dot{T}}{\partial X} = (K + \Psi)\,\frac{\partial \dot{e}}{\partial X} + \left\{K\,\frac{\partial (K + \Psi)}{\partial \widetilde{T}} + \Phi\,\frac{\partial K}{\partial \alpha}\right\}\dot{e}\,\frac{\partial e}{\partial X}, \tag{5.4}$$

where K, Φ and Ψ are given by (2.6) and (2.7). Then we have

$$\left[\frac{\partial \dot{T}}{\partial X}\right] = (K + \Psi)\left[\frac{\partial \dot{e}}{\partial X}\right] - \frac{1}{U^3}\left\{K\,\frac{\partial (K + \Psi)}{\partial \widetilde{T}} + \Phi\,\frac{\partial K}{\partial \alpha}\right\}a^2, \tag{5.5}$$

where (3.12), (3.13) and $[\dot{e}\ \partial e/\partial X] = [\dot{e}]\,[\partial e/\partial X]$ were used.

Substituting (5.3) and (5.5) into the relation (5.2), and referring to the propagation velocities (4.5) and (4.6), we have the *evolutional equation of the amplitude*

$$\frac{\delta a}{\delta t} = \Gamma\, a^2, \tag{5.6}$$

where,

$$\Gamma = 0, \tag{5.7a}$$

$$\Gamma = \frac{1 - c}{U_E}\;\frac{\widetilde{S}\left\{1 - \dfrac{\widetilde{S}^2}{N(\alpha)^2}\right\}}{M(\alpha)^2\left\{1 - \dfrac{(1 - c)\widetilde{S}^2}{M(\alpha)^2}\right\}^{\frac{1}{2}}}, \tag{5.7b}$$

for the unloading wave and for the loading wave, respectively, where $M(\alpha)' = \partial M(\alpha)/\partial\alpha$ and

$$N(\alpha) = \frac{M(\alpha)}{(1 + M(\alpha)\,M(\alpha)')^{\frac{1}{2}}}. \tag{5.8}$$

It is easy to solve equation (5.6), and we have

$$a(t) = a(0) \tag{5.9}$$

for the unloading wave and

$$a(t) = \frac{a(0)}{1 - a(0)\,\Gamma\, t} \tag{5.10}$$

for the loading wave, where $a(0)$ is the initial amplitude at $t = 0$. Then we can say that the unloading wave propagates with a constant amplitude while the loading wave may grow or decay.

If $a(0)\,\Gamma > 0$, then growth occurs and at the finite time

$$t_\infty = \frac{1}{a(0)\,\Gamma} \tag{5.11}$$

the amplitude of the acceleration wave tends to infinity and it coalesces into a *shock wave*. The time (5.11) is called the *blow time*, which can be expressed as

$$t_\infty = \frac{U_{\mathrm{E}}}{a(0)} \frac{M(\alpha)^2 \left\{1 - \dfrac{(1-c)\widetilde{S}^2}{M(\alpha)^2}\right\}^{\frac{1}{2}}}{\widetilde{S}\left\{1 - \dfrac{\widetilde{S}^2}{N(\alpha)^2}\right\}}\,. \tag{5.12}$$

The blow distance,

$$d_\infty = U\, t_\infty = \frac{U_{\mathrm{E}}}{a(0)} \frac{2\,M(\alpha)^2 \left\{1 - \dfrac{(1-c)\widetilde{S}^2}{M(\alpha)^2}\right\}}{\widetilde{S}\left\{1 - \dfrac{\widetilde{S}^2}{N(\alpha)^2}\right\}}\,, \tag{5.13}$$

denotes the distance before the wave blows up. The blow time and the blow distance are inversely proportional to the initial amplitude.

If $a(0)\,\Gamma < 0$, the wave decreases monotonically to zero as time tends to infinity.

From the yield translated stress (2.12), we may assume that

$$|\,\widetilde{S}\,| \leqq M(\alpha)\,(1 - M(\alpha)\,M(\alpha)')^{\frac{1}{2}}, \tag{5.14}$$

and then

$$1 \geqq 1 - \frac{(1-c)\widetilde{S}^2}{M(\alpha)^2} \geqq c + (1-c)\,M(\alpha)\,M(\alpha)' \geqq 0,$$

$$1 \geqq 1 - \frac{\widetilde{S}^2}{N(\alpha)^2} \geqq M(\alpha)\,M(\alpha)' \geqq 0.$$

Therefore Γ has the same sign as $\widetilde{S}$. Then we can say that in the tension state, that is, $\widetilde{T} > 0$, the compressive loading wave blows up while the expansive loading wave decreases; in the compression state, that is, $\widetilde{T} < 0$, the compressive loading wave decreases while the expansive loading wave blows up; and at $\widetilde{T} = 0$, the waves propagate with a constant amplitude.

For the perfectly plastic material with $a = c = 0$ we have

$$t_\infty = \frac{U_\mathrm{E}}{a(0)} \frac{M_0{}^2}{S\left\{1 - \frac{S^2}{M_0{}^2}\right\}^{\frac{1}{2}}}, \tag{5.15}$$

$$d_\infty = \frac{U_\mathrm{E}{}^2}{a(0)} \frac{M_0{}^2}{S}. \tag{5.16}$$

Substituting the yield stress (2.12) into (5.12) and (5.13) we have the blow time and the blow distance at the yield state

$$t_\infty = \frac{U_\mathrm{E}}{a(0)} \frac{1}{M(\alpha) M(\alpha)'^2} \left\{\frac{c + (1-c) M(\alpha) M(\alpha)'}{1 - M(\alpha) M(\alpha)'}\right\}^{\frac{1}{2}}, \tag{5.17}$$

$$d_\infty = \frac{U_\mathrm{E}{}^2}{a(0)} \frac{1}{M(\alpha) M(\alpha)'^2} \frac{c + (1-c) M(\alpha) M(\alpha)'}{(1 - M(\alpha) M(\alpha)')^{\frac{1}{2}}}. \tag{5.18}$$

They have usually very large values and for a material without isotropic work-hardening they are clearly infinite. For a perfectly plastic material the blow time is infinite but the blow distance tends to a constant value at the yield state,

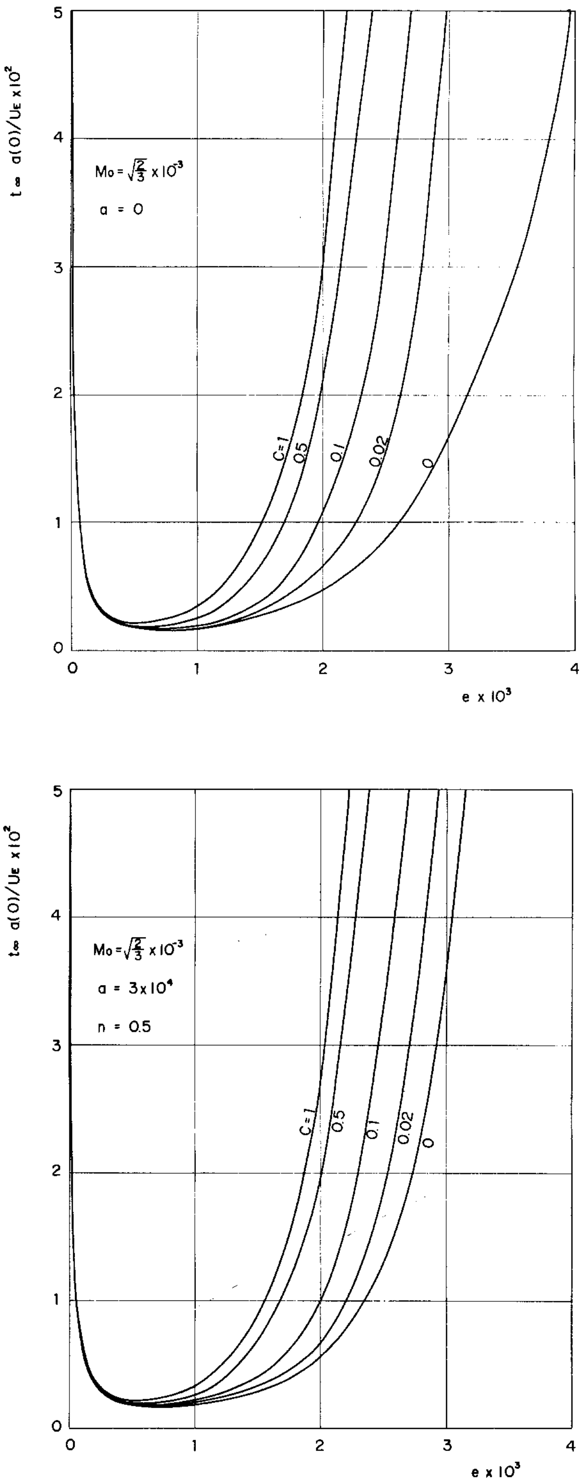
$$d_\infty = \frac{U_\mathrm{E}{}^2}{a(0)} M_0. \tag{5.19}$$

Figure 2 and Figure 3 show, respectively, the non-dimensional blow time $t_\infty a(0)/U_\mathrm{E}$ and the non-dimensional blow distance $d_\infty a(0)/U_\mathrm{E}{}^2$ against the strain in the loading state presented in Figure 2 of [10]. Clearly they are infinite at the stress-free state. The limiting blow distance (5.19) is marked by a black circle in Figure 3(a). The diagrams have minima between $e = 4 \times 10^2$ and $e = 10^3$, so we can say that the loading acceleration wave transforms into a shock wave in shortest time or distance when the material is loaded by nearly half of the yield stress.

Let us estimate the practical values of the blow time and the blow distance. For *a sinusoidal wave*

$$U_\mathrm{E} = \frac{\lambda_\mathrm{E}}{\tau}, \qquad a(0) = \frac{4\pi^2}{\tau^2} A(0), \tag{5.20}$$

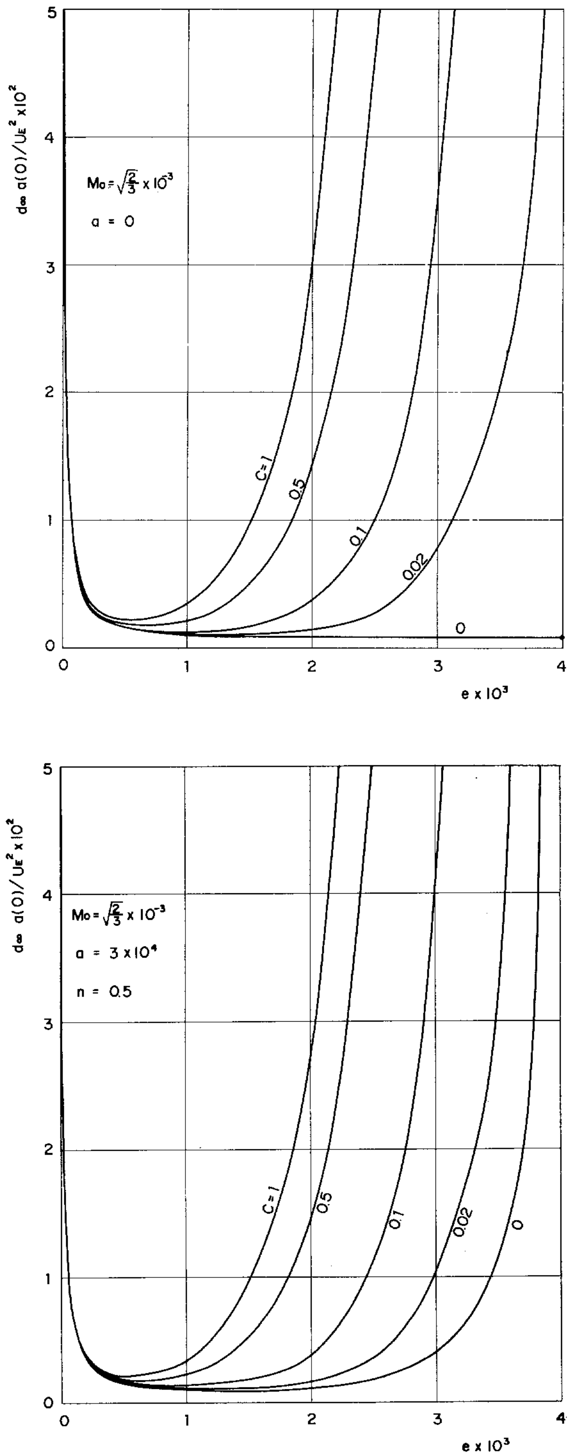
where λ_E is the wavelength of the unloading wave, τ is the period and $A(0)$ is the initial value of the amplitude of vibration of material particle. Then we have



(a)

(b)

Figure 2. Non-dimensional blow time against the strain in the loading state.



(a)

(b)

Figure 3. Non-dimensional blow distance against the strain in the loading state. The black circle denotes the limit value at the yield state.

$$\frac{U_E}{a(0)} = \frac{\lambda_E}{4\pi^2 A(0)} \tau, \tag{5.21}$$

$$\frac{U_E^2}{a(0)} = \frac{\lambda_E}{4\pi^2 A(0)} \lambda_E , \tag{5.22}$$

which indicates that the blow time and the blow distance are, respectively, given (in units of period and wavelength) by the multiples of the values plotted in Figs. 2 and 3 and $\lambda_E/\{4\pi^2 A(0)\}$.

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